Topoi for Photons: A Categorical Formulation of Electromagnetic Duality

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Abstract

We propose a categorical formulation of electromagnetic (Maxwell) duality by identifying a class of Grothendieck topoi (and suitable stacky sheaf models) whose internal cohomological data encode electric and magnetic sectors and admit a canonical duality isomorphism. Understanding this duality should ultimate improve our success-rate studying unobserved photonic laws. Building on the program of *topos potentials* and G-Theory – Maxwell correspondences, we formulate the *Toposic Maxwell Duality Conjecture* which asserts a natural equivalence between internal hypercohomology functors associated to dual gauge stacks. We provide precise definitions of the topos model $\mathscr{E}$ over a smooth spacetime manifold M, the gauge stack $\mathcal{G}$ encoding $U(1)$-connections (and higher analogues), and the pair of functors F, G selecting electric/magnetic sectors. We prove a partial result: on compact orientable surfaces (notably $M = T^2$) the conjectured duality reduces to Poincaré/Čech–de Rham duality and can be established up to the expected torsion and orientation twists. A worked toy computation on the two-torus exhibits the isomorphism of the relevant cohomological invariants. We conclude with numerical/heuristic checks (lattice discretizations and spectral invariants), physical implications for photonic systems, and explicit open problems, though we affirm the development of experiments, a variety of checks and the debate related to potential implications needs to broaden.

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1 Introduction

The physical observation of electromagnetic duality dates back to the classical Maxwell equations: in vacuum, electric and magnetic fields enter symmetrically after a Hodge duality with respect to a Lorentzian/Euclidean metric. In quantum and geometric contexts this symmetry manifests as electric–magnetic dualities, S-duality and in the interplay between line bundles and gerbes. The aim of our idea is to recast Maxwell duality entirely inside a Grothendieck topos (or appropriately stackified sheaf model) so that the duality becomes a *natural isomorphism* between cohomological functors internal to that topos. This is an unusual method, however it provides a conceptual bridge between Grothendieck/topos methods and photonic/Maxwell dualities, thereby extending previous work on topos potentials and G-Theory–Maxwell links. This method is currently not present, nor being used within the scientific community, nonetheless we believe this method has a lot of potential.

Our main output is a precise conjecture (the *Toposic Maxwell Duality Conjecture*) and a partial verification in the compact two-dimensional case. We need tools for this and the tools are categorical: stacks, adjoint functors, hypercohomology, and Čech–de Rham comparisons inside a topos. Physically, the toposic perspective clarifies when and how metric-dependent Hodge duals can be replaced by purely cohomological dualities in the presence of orientation and local system constituents.

Summary of results

- We define a class $\mathfrak{C}$ of topoi (and stacky sheaf models) over a smooth manifold M that encode local electromagnetic data and gauge information (§3).
- We formulate the *Toposic Maxwell Duality Conjecture* as a natural isomorphism of graded cohomology functors (Conjecture 4.1).
- We prove a special-case theorem for $M = T^2$ (Theorem 5.1) via the Čech–de Rham comparison and internal Poincaré duality (§5).
- We compute the toy example on the two-torus (§6) and outline numerical/spectral checks (§7).

2 Background

We now set the foundational stage by reviewing the necessary classical results that will be lifted into the toposic framework.

2.1 Grothendieck topoi and stacks

A Grothendieck topos $\mathscr{E}$ will be considered together with a geometric morphism $\pi : \mathscr{E} \to \mathrm{Sh}(M)$ where M is a smooth manifold (spacetime). We denote internal sheaves, stacks and their cohomology as objects of $\mathscr{E}$. A *stacky sheaf model* will mean a presentable higher stack internal to $\mathscr{E}$ obtained by stackification of a prestack of groupoids (or ∞-groupoids) describing local gauge data. We can now study formalism.

2.2 Maxwell formalism

Classically, Maxwell data on M can be encoded by a $U(1)$-connection A with curvature $F = dA$ (locally a closed 2-form). The gauge-invariant content of the electromagnetic field is the de Rham cohomology class $[F] \in H^2_{\mathrm{dR}}(M)$. Electromagnetic duality in an n-dimensional oriented Riemannian manifold is traditionally realized as a Hodge isomorphism

$$\star : \Omega^p(M) \xrightarrow{\simeq} \Omega^{n-p}(M), \tag{2.1}$$

which induces isomorphisms on cohomology when combined with Poincaré duality. Equation (2.1) will be replaced below by purely toposic cohomological isomorphisms. The question now is how we might further advance our model.

3 Model and definitions

We shall now state the categorical model precisely.

Definition 3.1 (Toposic Maxwell data)**.** *Let M be a smooth oriented n-manifold. A* toposic Maxwell model *consists of the following constituents:*

1. *A Grothendieck topos $\mathscr{E}$ together with a geometric morphism $\pi : \mathscr{E} \to \mathrm{Sh}(M)$.*
2. *A* gauge stack *$\mathcal{G}$ internal to $\mathscr{E}$ presenting $U(1)$-bundles with connection and higher analogues (gerbes, p-gerbes) as appropriate.*
3. *Two functors ("sectors") $F, G : \mathrm{St}(\mathscr{E}) \to \mathrm{GrAb}$ from stacks in $\mathscr{E}$ to graded abelian groups selecting electric and magnetic cohomological invariants respectively.*

The model's potential is fully realized only when working with hypercohomology.
We denote our internal hypercohomology by

$$\mathbb{H}^p(\mathscr{E}; \mathcal{A}) := \mathbf{R}^p\Gamma_{\mathscr{E}}(\mathcal{A}), \tag{3.1}$$

for $\mathcal{A}$ a complex of abelian sheaves/complexes in $\mathscr{E}$ and $\Gamma_{\mathscr{E}}$ the global-sections functor of $\mathscr{E}$. The functors F and G are required to factor through hypercohomology on canonical complexes:

$$F(\mathcal{G}) \cong \mathbb{H}^\bullet(\mathscr{E}; \mathcal{F}) \qquad G(\mathcal{G}) \cong \mathbb{H}^\bullet(\mathscr{E}; \mathcal{G}^\vee), \tag{3.2}$$

where $\mathcal{F}$ and $\mathcal{G}^\vee$ are complexes in $\mathscr{E}$ representing electric and magnetic fields (with $\mathcal{G}^\vee$ the internal dual complex of $\mathcal{F}$ in the derived category of $\mathscr{E}$).

Definition 3.2 (Admissible topoi class $\mathfrak{C}$)**.** *We say $\mathscr{E} \in \mathfrak{C}$ if:*

1. *$\mathscr{E}$ is locally connected and locally contractible over M.*
2. *The Čech hypercohomology of good coverings of M computes internal hypercohomology, i.e. a Čech–de Rham comparison holds internally.*
3. *Internal Poincaré duality holds for perfect complexes up to a canonical orientation twist.*

4 Our Main conjecture

We now state the principal conjecture in a precise functorial form.

Conjecture 4.1 (Toposic Maxwell Duality Conjecture)**.** *Let M be an oriented smooth n-manifold and let $\mathscr{E} \in \mathfrak{C}$ be an admissible topos over M with gauge stack $\mathcal{G}$. There exist canonical complexes $\mathcal{F}, \mathcal{F}^\vee$ in $D^b(\mathscr{E})$ (the bounded derived category of abelian sheaves in $\mathscr{E}$) and a natural isomorphism of graded functors*

$$\Phi_{\mathscr{E}} : \mathbb{H}^\bullet(\mathscr{E}; \mathcal{F}) \xrightarrow{\simeq} \mathbb{H}^{n-\bullet}(\mathscr{E}; \mathcal{F}^\vee) \otimes_{\mathbb{Z}} \mathrm{or}_{\mathscr{E}}, \tag{4.1}$$

where $\mathrm{or}_{\mathscr{E}}$ is the internal orientation local system of $\mathscr{E}$. The isomorphism (4.1) *is natural in geometric morphisms $f : \mathscr{E}' \to \mathscr{E}$ preserving the above structures, and recovers the classical metric Hodge duality on de Rham cohomology in the presence of a compatible metric and connection components.*

Remark 4.2. *The twist by* $\mathrm{or}_{\mathscr{E}}$ *in* (4.1) *encodes the usual orientation-dependence of Poincaré duality; when $\mathscr{E}$ is oriented and the coefficients are untwisted, the tensor factor may be dropped.*

5 Our Proof sketch and special-case theorem

Our work is, new work and hence we cannot presently prove Conjecture 4.1 in full generality. Instead we establish it for compact orientable surfaces (notably the 2-torus), where classical dualities and Čech–de Rham methods are fully available.

Theorem 5.1. *Let $M = T^2$ and let $\mathscr{E} = \mathrm{Sh}(T^2)$ be the ordinary sheaf topos (viewed as admissible). Let $\mathcal{F} = \Omega^\bullet_{\mathrm{dR}}$ (the de Rham complex) and let $\mathcal{F}^\vee$ be its derived dual. Then there is a canonical natural isomorphism*

$$\mathbb{H}^\bullet(\mathscr{E}; \Omega^\bullet_{\mathrm{dR}}) \xrightarrow{\simeq} \mathbb{H}^{2-\bullet}(\mathscr{E}; \Omega^\bullet_{\mathrm{dR}}). \tag{5.1}$$

In particular, the cohomological invariants of the electric and magnetic sectors coincide up to the degree shift $2 \mapsto 0$ and orientation twist which is trivial on T^2.

Proof sketch. The proof proceeds in three steps.

Step 1: Čech–de Rham comparison. For T^2 choose a good cover $\mathfrak{U} = \{U_i\}$ by contractible opens. The Čech bicomplex for $\Omega^\bullet$ computes de Rham hypercohomology. Concretely, there is a quasi-isomorphism

$$\check{C}^\bullet(\mathfrak{U}; \Omega^\bullet) \simeq \mathbf{R}\Gamma(T^2; \Omega^\bullet), \tag{5.2}$$

and this yields the classical de Rham theorem identifying hypercohomology with singular (or sheaf) cohomology with real coefficients.

Step 2: Poincaré duality on T^2. The torus is a compact oriented manifold of dimension 2; classical Poincaré duality gives an isomorphism

$$H^p_{\mathrm{dR}}(T^2) \xrightarrow{\simeq} H^{2-p}_{\mathrm{c}}(T^2)^\vee, \tag{5.3}$$

which (using nondegenerate pairings from integration of wedge-products) yields the desired degree-reversal isomorphism on the rather interesting hypercohomology groups.

Step 3: Stack/gauge interpretation. Passing to the gauge stack describing $U(1)$-connections (equivalently, to degree 1 Deligne complexes) the above identifications extend: curvature classes live in $H^2_{\mathrm{dR}}(T^2)$ while holonomy-type invariants live in $H^1_{\mathrm{dR}}(T^2; \mathbb{R}/\mathbb{Z})$. On T^2 these groups are finite-rank and dual under cup-product and integration. Hence the functors F, G of (3.2) agree up to the degree reversal predicted in (5.1), establishing (5.1). □

6 Toy example: computations on the two-torus

We present explicit computations to illustrate Theorem 5.1.

6.1 Classical cohomology computations

For T^2 we have

$$H^0_{\mathrm{dR}}(T^2) \cong \mathbb{R}, \qquad H^1_{\mathrm{dR}}(T^2) \cong \mathbb{R}^2, \qquad H^2_{\mathrm{dR}}(T^2) \cong \mathbb{R}. \tag{6.1}$$

These groups may be computed by representatives: constant functions, the two independent 1-forms dx, dy, and the volume form $dx \wedge dy$. The cup-product pairing

$$H^1_{\mathrm{dR}}(T^2) \times H^1_{\mathrm{dR}}(T^2) \xrightarrow{\smile} H^2_{\mathrm{dR}}(T^2) \xrightarrow{\int_{T^2}} \mathbb{R} \tag{6.2}$$

is nondegenerate and manifests Poincaré duality on the torus. Equation (6.2) is the cohomological incarnation of electric/magnetic pairing.

6.2 Toposic interpretation

Interpreting the torus cohomology inside $\mathscr{E} = \mathrm{Sh}(T^2)$ (no higher stackification necessary) we set $\mathcal{F} = \Omega^\bullet_{\mathrm{dR}}$ and observe the canonical identification (5.1). The electric sector classifying curvature 2-forms and the magnetic sector classifying holonomy data in H^1 are therefore dual: H^2 classes pair with H^0 (background charges) and H^1 pairs with itself as in (6.2).

7 Numerical / experimental tests and process

There is a need for an extensive process, in order to connect the toposic conjecture to numerics and experiments we hence propose the following strategies.

Discrete lattice toy model. Replace M by a lattice approximation M_h and replace sheaf cohomology with cohomology of discrete chain complexes (cellular cohomology). Compute discrete analogues of $\mathbb{H}^\bullet(\mathscr{E}; \mathcal{F})$ and verify the degree-reversal isomorphism up to finite-size and boundary corrections. One expects rapid convergence of spectral gaps in the Laplace operator to continuum values.

Spectral invariants. Formulate the duality in terms of spectral data of the Laplace–de Rham operator Δ_p acting on p-forms inside the topos-realization. Numerically compare spectra $\mathrm{Spec}(\Delta_p)$ and $\mathrm{Spec}(\Delta_{n-p})$; suitably regularized determinants should be related by the duality isomorphism.

Physical signatures. In photonic crystals or metamaterials, electromagnetic dualities manifest as symmetry of mode spectra under duality transformations. The toposic viewpoint suggests searching for invariants (e.g. integrated Chern numbers or flux quantization conditions) that are stable under stacky deformations and measurable via scattering experiments. We would like to encourage researchers to further investigate this process.

8 Foundations of Topos Potentials

We will now cover *Topos Potentials* since these motivate the toposic Maxwell program.

8.1 Local-to-global potentials

A topos potential is an internal object in $\mathscr{E}$ which encodes local potential data (connections, higher potentials) together with gluing cocycles that represent global gauge information. Concretely, for $U(1)$-connections one models the potential as a degree-1 Deligne-type complex internal to $\mathscr{E}$; higher potentials (gerbes) are modeled by higher-degree Deligne complexes. The strength of the topos viewpoint is that these complexes and their hypercohomology are built inside the same ambient logic, permitting natural internal manipulations (pullback, pushforward, dualization) without leaving $\mathscr{E}$.

8.2 Internal descent and gauge transformations

Descent for potentials is internalized: local gauge transformations and higher homotopies are morphisms in the slice 2-category of stacks over $\mathscr{E}$. This internal descent viewpoint clarifies obstructions (cocycles that fail to be coboundaries) as internal cohomology classes and makes manifest the functoriality of gauge data under geometric morphisms of topoi.

8.3 Consequences for physical modeling

From a modeling perspective, topos potentials allow one to:

- treat metric-independent features (flux quantization, holonomy) as intrinsic to the theory;
- manage interface and defect data uniformly by viewing them as boundary conditions in the ambient topos;
- define families of potentials parametrized by external moduli (material parameters) as objects in internal mapping stacks.

9 G-Theory and Maxwell Correspondences

We will now look into how a G-Theory viewpoint organizes correspondences between gauge-theoretic invariants and Maxwell-type dualities.

9.1 G-Theory perspective

G-Theory here refers to a (informal) organizing principle that packages gauge data (bundles, gerbes, connections) and their cohomological invariants in a single algebraic framework amenable to categorical dualities. From the toposic angle, G-Theory is represented by the derived category of perfect complexes on the gauge stack together with natural pairings (cup-product, trace maps) that realize physical pairings between electric and magnetic sectors.

9.2 Correspondence diagram

Fundamental correspondences take the schematic form

$$\text{Gauge stack (potentials)} \Longrightarrow \text{Cohomology complexes} \longrightarrow \text{Spectral / Dual complexes}$$

where the left arrow encodes passage from geometric gauge data to complexes computing hypercohomology, and the right arrow is a derived duality (Poincaré-type pairing) implementing Maxwell exchange. We realize this is an experimental approach.

9.3 Practical consequences

Viewing experimental Maxwell duality as a G-Theory correspondence highlights:

- the role of derived functors (Ext, **R**Hom) in constructing dual invariants;
- how torsion phenomena and integral lifts obstruct naive dualizations (explaining quantization effects);
- routes to generalize to nonabelian or higher-form gauge theories by replacing perfect complexes with appropriate derived enhancements.

10 Stacky Sheaves and Cohomological Dualities

Let us emphasize structural features of stacky sheaves that are essential for the toposic duality.

10.1 Higher-stack organization of fields

Fields (connections, gerbe-potentials) naturally live in higher stacks rather than ordinary sheaves: automorphisms, higher automorphisms and homotopies are part of the structure. The derived hypercohomology of these stacks captures both classical invariants and higher obstructions; importantly, pushforward and pullback along geometric morphisms preserve this homotopical structure when performed in the derived sense.

10.2 Duality functors

Given a perfect complex $\mathcal{F}$ on a stacky site, its derived dual $\mathcal{F}^\vee = \mathbf{R}\mathcal{H}om(\mathcal{F}, \mathbb{Z})$ yields a natural candidate for the magnetic complex when $\mathcal{F}$ models the electric sector. The cohomological duality is then a manifestation of the canonical pairing

$$\mathbf{R}\Gamma(\mathcal{F}) \otimes^{\mathbf{L}} \mathbf{R}\Gamma(\mathcal{F}^\vee) \to \mathbf{R}\Gamma(\mathbb{Z}) \xrightarrow{\int} \mathbb{Z},$$

internalized to the topos. Compatibility of these pairings with descent and with geometric morphisms is the crux of making Conjecture 4.1 natural.

10.3 Remarks on nonabelian and higher generalizations

Although our abstract indeed focuses on abelian phenomena (Maxwell duality, gerbes), the stacky-sheaf apparatus is well-suited to nonabelian generalizations: replace abelian complexes by Lie-algebroid-type objects or by derived stacks of local systems; dualities in those contexts are subtler but can be approached via similar derived pairing techniques, often requiring additional data (e.g. shifted symplectic structures). These will need to be studied.

11 Discussion and future directions

11.1 Physics implications

A toposic reformulation of Maxwell duality abstracts away the metric-dependent Hodge operator and replaces it by structural cohomological isomorphisms. This has several potential implications:

- **Background independence.** Duality becomes visible even in regimes where no global metric is present (e.g. topologically nontrivial photonic media), so long as the topos admits the Čech–de Rham comparison and orientation data.
- **Higher charges and gerbes.** The present formalism extends naturally to p-form gauge fields and their duals (gerbes), since stacky sheaf models of higher $U(1)$-bundles are built into the topos framework.

- **Quantization and torsion.** Torsion classes and discrete fluxes appear naturally as obstructions to lifting the toposic isomorphism to integral coefficients; this clarifies quantization conditions in a purely categorical setting.

11.2 A good many open problems

We list only three of the concrete problems we have encountered, in order to advance this program and we leave our work open.

1. **Higher-dimensional verification.** Prove Conjecture 4.1 for compact orientable 4-manifolds M with nontrivial torsion. This will require a refined treatment of orientation-twists and integral lifts.
2. **Stacky S-duality.** Extend the conjecture to nonabelian gauge stacks (e.g. $SL(2, \mathbb{Z})$-twisted families) and relate to known S-duality statements in string theory.
3. **Analytic spectral formulation.** Produce a rigorous spectral invariant that is preserved under the toposic duality and can be computed numerically for photonic materials.

12 Conclusion

We have proposed a precise categorical conjecture that identifies a class of topoi for which Maxwell duality is realized as an internal cohomological equivalence. A partial verification on T^2 and explicit computations support the plausibility of the conjecture. The toposic viewpoint opens avenues for metric-independent formulations of electromagnetic dualities, higher-form generalizations, and computational tests in discretized and photonic systems. Studying photonic law is an undertaking best taken one small step at the time.